# Classical solutions of $\mathbb{C}P^n$ non linear $\sigma$ -models; an algebraic geometrical description*

R. CATENACCI

Dipartimento di Matematica dell'Università
Via Strada Nuova 65, Pavia (Italy)
INFN, Section of Pavia

M. CORNALBA

Dipartimento di Matematica dell'Università
Via Strada Nuova 65, Pavia (Italy)

C. REINA

Dipartimento di Fisica dell'Università
Via Celoria 16, Milano (Italy)

**Abstract.** *The classical SO (3)-invariant σ-model and its suitably generalized versions are studied from the geometrical point of view. Known mathematical results concerning harmonic and holomorphic maps of a Riemann surface into the n-dimensional complex projective space are briefly reviewed. These are used to give a classification of classical solutions (both instantons and a certain subclass of unstable solutions) of* $\mathbb{C}P^n$ *models and to study the properties of their energy spectrum.*

## 1. INTRODUCTION

1.1. In recent years, several theories and models have been proposed in physics, which were realized to be rich in geometrical meaning. Among these, some of the most important to physics seem to be at present the Yang-Mills gauge theories, for which the dynamical field can be regarded as a connection on a principal fibre bundle [1].

For Yang-Mills fields, this geometrical interpretation has been far from a mere

(*) Work partially supported by Gruppo Nazionale di Fisica Matematica, C.N.R. and Gruppo Nazionale per le Strutture Algebriche e Geometriche e Applicazioni, C.N.R.

translation of the problem into a more or less sophysticated mathematical form, yielding instead a deep insight into the theory and providing detailed prescriptions for constructing and classifying classical solutions of the field equations [2]. This has been particularly the case for the elliptic version of the field equations, i.e. for the study of Yang-Mills instantons, which has been shown [3] to be equivalent to a problem of complex analysis and finally to one of algebraic geometry. Indeed, it was by means of the powerful tools of modern algebraic geometry that the problem was finally solved [4].

Complex and algebraic geometrical methods have been recently applied to some other differential equations of mathematical physics. Besides the Yang--Mills instantons recalled above, there is the more general programme of twistor theory [5], some applications to gravitational instantons [6, 7] and the study of completely integrable dynamical systems such as the Korteweg - de Vries equation [8]. Finally, Hitchin [9] has shown that the Bogomolny equation for non-abelian monopoles can be treated as well in complex geometrical terms.

In this paper we describe in full details a further application of methods from algebraic geometry to physics, that is we study the classical solutions of the 2-dimensional $\mathbb{C}P^n$ models. Although this may be considered a fairly simple exercise, compared with the results quoted above, it seems interesting to explore this problem, since it is in some sense a low dimensional analogue of the Yang--Mills theory. Besides, it may help in understanding how some of the basic tools of algebraic geometry work, when applied to a problem of mathematical physics.

1.2. The physical interest of instanton solutions was first discovered in the quantum theory of Yang-Mills fields [10]. From the classical point of view the basic feature of instantons is that they provide absolute minima for the action functional of the theory, among fields with a given topological structure (depending on boundary conditions).

A similar phenomenon was shown [11] to occour in 2-dimensional ferromagnets, when described by means of the standard $SO(3)$ - invariant $\sigma$-model. This model was later generalized to $SU(n+1)/\mathbb{Z}_{n+1}$-invariant $\sigma$-models, which are now called $\mathbb{C}P^n$ models. It was immediately noted [13] that the general $\mathbb{C}P^n$--instanton solution on $S^2$ could be expressed in terms of relational functions of one complex variable. Since these are algebraic objects on the Riemann sphere, one may think as a «must» to understand also the $\mathbb{C}P^n$ models in algebraic geometrical terms. As we shall see, this will lead to a rather complete classification of finite energy solutions subjected to generalized boundary conditions and to a detailed description of the parameter space on which such instanton solutions depend. Some of these results have been already published in a series of papers [14 - 34 - 36], while this paper is devoted to collect them in a detailed introductory way.

1.3. In section 2 we shall recall the structure of the basic non linear $\sigma$-model and of its generalizations. In particular various natural «parametrizations» of the field, as well as suitable generalized boundary conditions, will be discussed. Section 3 deals with the translation of the problem into a form in which the natural underlying complex structures are made, apparent. This is by no means a mathematical trick, since such structures do actually enter in a canonical way into the $\mathbb{C}P^n$ models and one cannot effort to neglect them, unless one is willing to take the hard way in solving the field equations. Indeed, there are plenty of results, already available from the mathematical literature, which are relevant for the solution of the problem. Some of these will be reviewed in section 3.

In the next section, we shall use the power of complex method to yield actual classical solutions of $\mathbb{C}P^n$ models with generalized boundary conditions. No differential equations, other than Cauchy-Riemann equations, will enter this section. The results by Din and Zakrzewski [15], Burns, Glaser and Stora, which have been fully proved and generalized by Eells and Wood [16], will be briefly recalled and adopted to present a family of classical solutions which turn out to be unstable. Finally, we shall deal with the classical observables - i.e. the energy and the «topological charge» - in a synthetic way, which avoids cumbersome computations aimed at obtaining actual analytical expressions for the solutions.

We shall then enter the mathematical core underlying the physical problem. First, in section 5, we shall give a brief review of the results available from algebraic geometry which will be employed in the sequel. Basic an specialized references will be given. In section 6 we shall describe, as far as it is possible at the present time, the energy spectrum and the parameter spaces of classical solutions of $\mathbb{C}P^n$ models. Generalized boundary conditions will be considered, as well as the standard boundary conditions; in the former case less complete but interesting results will be presented.

**Acknowledgements**

We thank the Editorial Board of «Geometry and Physics» for inviting us to write this paper, after a short note [37] was presented by one of us at the meeting «Geometry and Physics» held in Florence in september 1982.

## 2. $CP^n$ NON LINEAR $\sigma$-MODELS

2.1. In the simplest non linear $\sigma$-model, the dynamical field is a map $\phi : \mathbb{R}^2 \to S^2$.

The field $\phi$ is usually represented by a vector $\underline{n} \in \mathbb{R}^3$, satisphying the constraint $\underline{n} \cdot \underline{n} = 1$. If, however, $S^2$ is idenfied with the Riemann sphere $\mathbb{P}^1$ (hereinafter we shall drop the prefix $\mathbb{C}$, since no confusion can arise), a natural complex parametrization for the field $\phi$ can be given [17]. In this case, the energy functional takes the form

$$E(\phi) = \frac{1}{2} \int_{\mathbb{R}^2} a^{ij} h_{\alpha\bar{\beta}} \partial_i \phi^\alpha \partial_j \phi^\beta v(a),$$

where a is the Euclidean metric on $\mathbb{R}^2$, $v(a)$ is its volume element and $h$ is the Fubini-Study metric on $\mathbb{P}^1$ (see Appendix A). Classical solutions for this model with finite energy are local extrema of $E(\ )$, subjected to the boundary condition $\phi \to$ const for $|x| \to \infty$. This boundary condition makes it possible to extend $\phi$ continously to the one point compactification of the domain $\mathbb{R}^2 \cup \{\infty\} = S^2 = \mathbb{P}^1$; the facts that $\phi$ has finite energy *and* is an extremal of $E$ imply that this extension $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is in fact smooth. Thus one can restrict the search for classical solutions to smooth maps of $\mathbb{P}^1$ into itself.

2.2. This model can be generalized as follows. First note that the compactification of the domain into $S^2$ is not the only possible choice leading to finite energy solutions. Indeed, if we allow ourselves to change the metric on $\mathbb{R}^2$, we may envisage periodicity conditions on the map (and on the metric) which make it possible to translate the problem into one for maps into $\mathbb{P}^1$ from compact two--dimensional orientable surfaces other than $S^2$. An alternative, and more intutive, way of obtaining the same «generalization» is to imagine oneself studying the Heisemberg model of a two-dimensional ferromagnet more complex than a plane sheet, as a torus or a compact surface with $g$ «holes» can be (1).

A further natural generalization [12, 13] is to study maps into higher dimensional projective spaces $\mathbb{P}^r$ (see Appendix A, for a short account on these manifolds). This is nice from the physical point of view, since such models exhibit a $SU(r+1)$ internal symmetry group (2).

2.3. In order to define the energy functional for these generalized cases, it is

(1) To avoid further mathematical complications, we shall not discuss here surfaces with boundary (as a disk), which might as well be interesting from the phisical point of view.

(2) Actually, the effective internal symmetry group is $SU(r+1)/\mathbb{Z}_{r+1}$. For instance, in the case $r = 1$, we recover the basic $\sigma$-model with symmetry group $SU(2)/\mathbb{Z}_2 \simeq SO(3)$.

convenient to assume that the domain of the problem is orientable. As already mentioned, it is simpler to limit oneself to connected, compact surfaces without boundary. It is well known that diffeomorphism classes of such surfaces are classified by a non-negative integer $g$, called the *genus* of the surface [18]. The genus and the Euler-Poincarè characteristic $\chi$ of the surface are related by $\chi = 2 - 2g$. This shows that a surface $S_g$ of genus $g$ is diffeomorphic to a «sphere with $g$ handles».

Let now

$$\phi : S_g \to \mathbb{P}^r$$

be a smooth map. Composing with coordinates, one can locally represent $\phi$ as

$$(x^1, x^2) \xrightarrow{\phi} \xi^i(x^1, x^2),$$

where $x^\mu$, $\xi^i$ are local coordinates on $S_g$ and $\mathbb{P}^r$ respectively ($\mu = 1, 2; i = = 1, \ldots, r$). Consider now the 1-forms

$$\mathrm{d}\xi^i = \frac{\partial \xi^i}{\partial x^\mu}\,\mathrm{d}x^\mu, \quad \mathrm{d}\xi^{\bar{i}} = \frac{\partial \overline{\xi^i}}{\partial x^\mu}\,\mathrm{d}x^\mu,$$

which can be considered as 1-forms on $S_g$, with values in the holomorphic ($T^{(1,0)}$) and the antiholomorphic ($T^{(0,1)}$) tangent bundle of $\mathbb{P}^r$. For any metric $a = a_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ given on $S_g$, one can construct a *global* $T^{(1,0)} \otimes T^{(0,1)}$ valued 2-form on $S_g$, given by

$$\epsilon^{i\bar{j}} = \mathrm{d}\xi^i \wedge \mathrm{d}\xi^{\bar{j}} = \sqrt{\det a_{\mu\nu}}\, a^{\mu\nu} \partial_\mu \xi^i\, \partial_\nu \xi^{\bar{j}}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2.$$

In order to produce a global section of $\wedge^2(S_g)$, we can use the Fubini Study metric $h$ of $\mathbb{P}^r$ (see Appendix A), yielding

$$\epsilon(\phi) = h_{i\bar{j}}\,\epsilon^{i\bar{j}}(\phi).$$

The integral

$$E(\phi) = \int_{S_g} \epsilon(\phi)$$

will be called the *energy* (3) of the $\mathbb{P}^r$ model. Stationary energy maps, i.e. maps $\phi : S_g \to \mathbb{P}^r$ satisfying the Euler-Lagrange equations for the functional $E$, will

(3) Since the domain of the problem is Riemannian and the field equations arising from the variational problem $\delta E = 0$ are elliptic, $E(\phi)$ will be called the energy of the field $\phi$ instead of the action of $\phi$.

be called classical solutions. In the following we shall limit ourselves to study classical solutions of finite energy. Those which give local minima of $E$, i.e. stable classical solutions of finite energy, will be called instantons.

2.3. We come now to discuss some parametrizations one can give to the maps $\phi$. First of all, one can use homogeneous coordinates on $\mathbb{P}^r$, representing $\phi$ as a set $z^0(x_1, x_2), \ldots, z^r(x_1, x_2)$. In this case the energy takes the form

$$E(\phi) = \int_{S_g} v(a) \, \frac{(z^i \overline{z}_i) \partial_\mu z^j \partial^\mu \overline{z}_j - \overline{z}_k z^j \partial_\mu z^k \partial^\mu \overline{z}_j}{(z^i \overline{z}_i)^2} \, .$$

One can easily verify that this action is invariant under local $\mathbb{C}^*$ gauge transformations, given by $z^i \to \lambda(x) z^i$, where $\lambda(x)$ is any non-vanishing complex valued function of $x_1, x_2$.

Restricting oneself to the $S^{2r+1}$ sphere given by $z^i \overline{z}_i = 1$, one has the following from for the energy,

$$\begin{cases} E(\phi) = \displaystyle\int_{S_g} v(a) \, (\partial_\mu z^j \partial^\mu \overline{z}_j - \overline{z}_k \partial_\mu z^k \, z^j \partial^\mu \overline{z}_j) \\ z^i \overline{z}_i = 1 \, , \end{cases}$$

which is invariant under local $U(1)$ gauge transformations given by $z^k \to e^{i\chi(x)} z^k$. This can be made explicitly apparent writing $E(\phi)$ in a gauge invariant form;

$$E(\phi) = \int_{S_g} D_\mu z^i D^\mu \overline{z}_i ,$$

$$D_\mu = \partial_\mu + i \, A_\mu ,$$

$$A_\mu = \frac{i}{2} \, \{ \overline{z}_k \partial_\mu z^k - z^k \partial_\mu \overline{z}_k \} ,$$

$$z^k \overline{z}_k = 1 .$$

Here $iA_\mu dx^\mu$ is a $U(1)$ connection on the principal bundle $S^{2r-1} \xrightarrow{U(1)} \mathbb{P}^r$ (see Appendix A), playing the rôle of a gauge potential. This is the form in which $\mathbb{P}^r$ models have been introduced in the physical literature. Note, however, that the connection $iA_\mu dx^\mu$ does not enter in a dynamical way in the energy functional, but has to be considered as an «esternal field» transforming in such a way that the local $U(1)$ invariance of the problem is preserved. We see then from the very beginning that, if $z^k(x^1, x^2)$ is the representation of a solution $\phi$ of the variational problem $\delta E = 0$, so is $e^{i\chi(x^1,x^2)} z^k$, for any real function $\chi(x^1, x^2)$.

In the following we shall use the parametrization naturally given in terms of the local coordinates $\xi^k$ on $\mathbb{P}^r$. This has the advantage that there are no more costraints on the field variables, nor non dynamical degrees of freedom. Finally we note that the internal (global) symmetry group is reduced in this case to $SU(r+1)/\mathbb{Z}_{r+1}$, which acts effectively as the isometry group of the Fubini--Study metric on $\mathbb{P}^r$. When this parametrization is assumed, the energy functional takes the form

$$E(\phi) = \int_{S_g} v(a) \; \frac{(1 + \xi^k \overline{\xi}_k)\, \partial^\mu \xi^j \partial_\mu \overline{\xi}_j - \overline{\xi}_k \partial^\mu \xi^k \, \xi^j \partial_\mu \overline{\xi}_j}{(1 + \xi^k \overline{\xi}_k)^2} \; .$$

2.4. For any $C^\infty$ map $\phi : S_g \to \mathbb{P}^r$, we define the *tension field* [19] of $\phi$ by the following local expression

$$\tau(\phi)^i = a^{\mu\nu} \left( \frac{\partial^2 \xi^i}{\partial x^\mu \partial x^\nu} - \Gamma^\lambda_{\mu\nu} \frac{\partial \xi^i}{\partial x^\lambda} + \gamma^i_{jk} \frac{\partial \xi^k}{\partial x^\mu} \frac{\partial \xi^j}{\partial x^\nu} \right),$$

where $\Gamma$ and $\gamma$ are the Christoffel symbols of the metric $a$ on $S_g$ and of the Fubini--Study metric $h$ of $\mathbb{P}^r$ respectively. It can be shown that $\tau(\phi)^i$ is a $C^\infty$ section of the bundle $\phi^* T\mathbb{P}^r$ over $S_g$, obtained by pulling back the tangent bundle $T\mathbb{P}^r$ via the map $\phi$. A map $\phi$ such that $\tau(\phi) = 0$ is called *harmonic*.

Recall now the

PROPOSITION 2.5. *A $C^\infty$ map $\phi : S_g \to \mathbb{P}^r$ is a critical point of the energy functional E, if and only if it is harmonic.*

*Proof.* By computing the first variation of $E$, one easily shows that it vanishes at $\phi$ if and only if the tension of $\phi$ vanishes, i.e. $\tau$ is the Euler-Lagrange operator of $E$. For a detailed proof see e.g. Eells and Sampson [20].

REMARK. The proposition above gives us the field equations for the $\mathbb{P}^r$ models in the form $\tau(\phi)^i = 0$. Note that this is true in general for any map $\phi : X \to Y$, with $X$ compact, $Y$ complete.

## 3. CONFORMAL INVARIANCE AND COMPLEXIFICATION OF THE DOMAIN

Harmonic maps can be defined between any two Riemannian manidolfds $X$ and

$Y$, by means of the natural generalization of the energy functional introduced above. We recall now a result which is particularly interesting for the present case.

PROPOSITION 3.1. *A map $\phi : X \to Y$, which is harmonic with respect to a given metric structure a on $X$, is also harmonic with respect to any conformally equivalent metric structure $a' = f^2 \cdot a$ ($f^2$ smooth and positve) if and only if* $\dim X = 2$.

*Proof.* Let $\tau'(\phi)$ be the tension field of $\phi$ with respect to $a'$. Then, if $\dim X = n$

$$\tau'(\phi)^i = \frac{1}{f^n} \frac{\partial}{\partial x^\mu}\left( f^n \sqrt{\det a^{\mu\nu}} \frac{a^{\mu\nu}}{f^2} \frac{\partial y^i}{\partial x^\nu}\right) +$$

$$+ \frac{a^{\mu\nu}}{f^2} L^i_{jk} \frac{\partial y^j}{\partial x^\mu} \frac{\partial y^k}{\partial x^\nu},$$

where $x^\mu$, $y^i$ are local coordinates on $X$, $Y$ and $L$ are the Christoffel symbols of the metric on $Y$. At this point it suffices to notice that

$$\tau'(\phi)^i = \frac{1}{f^2} \tau(\phi)^i + (n-2)\sqrt{\det a^{\mu\nu}} \frac{a^{\mu\nu}}{f^2} \frac{\partial f}{\partial x^\mu} \frac{\partial y^i}{\partial x^\nu}.$$

REMARK. Conformal invariance is the reason why in the physical literature $\mathbb{P}^r$ models are mainly studied on a two-dimensional domain.

3.2. It is well known that any orientable surface, such as our surfaces $S_g$, admits a complex structure. It is therefore natural to give a version of the $\mathbb{P}^r$ models in which the power of complex methods can be used.

As a corollary to proposition 3.1, we see that in our case only a conformal class of metrics needs to be given on $S_g$. It is a classical result of Riemann's that there exists a one-to-one correspondence between conformal and complex structures on $S_g$ (see Appendix B, for an elementary account).

In order to give a definite meaning to a $\mathbb{P}^r$ model, one must be able to find out from the physics of the problem the conformal structure or, equivalently, the complex structure on $S_g$ one is dealing with. In the case of $\mathbb{P}^1$ models, which are physically interpretable as models for ferromagnetism on an actual surface of genus $g$ living in the physical space $\mathbb{R}^3$, the restriction of the standard Euclidean metric of $\mathbb{R}^3$ to $S_g$ defines unambigously a «physical» metric on $S_g$. This, in turn, selects a conformal structure and hence a unique complex structure on the domain of the model $S_g$. For $\mathbb{P}^r$ models, this «physical» interpretation is lacking. However, proposition 3.1 requires that a conformal structure on $S_g$ is given. Had this not been the case, one would not be able to define what is meant by harmonicity of a map $\phi : S_g \to \mathbf{P}^r$.

Finally we note that in any case, once a conformal structure on $S_g$ has been fixed, the corresponding complex structure turns $S_g$ into a complex one-dimensional compact manifold. In the following $S_g$ with a fixed complex structure will be called tout-court a (complex) curve and will be denoted by $C$. It is a standard result (see e.g. [21]) that any such curve, being compact, is algebraic, that is it can be realized as the zero locus of a finite number of homogeneous polynomials in $P^n$, for some $n$.

3.3. We shall now benefit of the fact that $C$ has a complex structure to give a new formulation of the $\mathbf{P}^r$ models. Let $z$ be a local complex coordinate on $C$. One-forms over $C$ can be split into $(1, 0)$ and $(0, 1)$ parts according to

$$\omega = \omega_\mu \mathrm{d}x^\mu = \frac{1}{2}(\omega_1 - i\omega_2)\,\mathrm{d}z + \frac{1}{2}(\omega_1 + i\omega_2)\,\mathrm{d}\overline{z}.$$

There is therefore a canonical way of splitting the forms $\mathrm{d}\xi^k$, $\mathrm{d}\overline{\xi}_k$ of section 2.4 as follows

$$\mathrm{d}\xi^k = \xi^k_z\,\mathrm{d}z + \xi^k_{\overline{z}}\,\mathrm{d}\overline{z},$$

$$\mathrm{d}\overline{\xi}^k = \overline{\xi}^k_z\,\mathrm{d}z + \overline{\xi}^k_{\overline{z}}\,\mathrm{d}\overline{z},$$

where the suffix $z$ $(\overline{z})$ denotes partial derivation with respect to $z$ $(\overline{z})$. Note that (dropping the index $k$)

$$\xi_z \mathrm{d}z \in T^{(1,0)}\mathbf{P}^r \otimes T^{*(1,0)}C,$$

$$\xi_{\overline{z}} \mathrm{d}\overline{z} \in T^{(1,0)}\mathbf{P}^r \otimes T^{*(0,1)}C,$$

$$\overline{\xi}_z \mathrm{d}z \in T^{(0,1)}\mathbf{P}^r \otimes T^{*(1,0)}C,$$

$$\overline{\xi}_{\overline{z}} \mathrm{d}\overline{z} \in T^{(0,1)}\mathbf{P}^r \otimes T^{*(0,1)}C.$$

By means of the Fubini-Study metric $h$ on $\mathbf{P}^r$, we can now construct two global real valued 2-forms on $C$, which are positive definite and locally given by

$$\epsilon^{(1,0)} = \frac{i}{2}\,h(\xi_z, \overline{\xi}_{\overline{z}})\,\mathrm{d}z \wedge \mathrm{d}\overline{z},$$

$$\epsilon^{(0,1)} = \frac{i}{2}\,h(\overline{\xi}_z, \xi_{\overline{z}})\,\mathrm{d}z \wedge \mathrm{d}\overline{z}.$$

Note that these forms depend only on the complex structure of $C$ and do not require any choice of a particular metric structure within the conformal class of metrics compatible with the complex structure of $C$. This immediately shows that $\epsilon^{(1,0)}$ and $\epsilon^{(0,1)}$ are conformally invariant.

It is now trivial to check that the energy of $\mathbb{P}^r$ models given in section 2.4 can be expressed as follows

$$E(\phi) = \int_C \epsilon^{(1,0)} + \int_C \epsilon^{(0,1)}.$$

In particular, the conformal invariance of $E$ is now apparent.

3.4. Consider now the difference $\int_C \epsilon^{(1,0)} - \int_C \epsilon^{(0,1)}$. If $\Phi = i\, h_{j\bar{k}} \mathrm{d}\xi^j \wedge \mathrm{d}\bar{\xi}^k$ is the Kähler form of $\mathbb{P}^r$ (see appendix A), one has;

$$\int_C \epsilon^{(1,0)} - \int_C \epsilon^{(0,1)} = i \int_C h(\xi_z, \bar{\xi}_{\bar{z}})\, \mathrm{d}z \wedge \mathrm{d}\bar{z} - i \int_C h(\bar{\xi}_z, \xi_{\bar{z}})\, \mathrm{d}z \wedge \mathrm{d}\bar{z} =$$

$$= i \int_C \phi^*(h_{j\bar{k}} \mathrm{d}\xi^j \wedge \mathrm{d}\bar{\xi}^k) = \int_C \phi^*(\Phi),$$

where $\phi^*$ denotes the pull-back map associated to $\phi$. The integral $\int_C \phi^*(\Phi)$ is classically called the *degree* of the map $\phi : C \to \mathbb{P}^r$; it is an integer, which will be denoted by deg $(\phi)$.

What matters in the following is that deg $(\phi)$ depends only on the homotopy class of $\phi$. In fact, in integral cohomology, one has that the pull-back $\phi^*[H]$ of the cohomology class of a hyperplane $H$ in $P^r$ is given by $\phi^*[H] = \deg(\phi)\eta$, where $\eta$ is the (positively oriented) generator of $H^2(C, \mathbb{Z})$ (4).

(4) An elementary proof of the homotopy invariance of deg $(\phi)$ can be obtained as follows. Let $F_t : C \times [0,1] \to P^r$ be a smooth homotopy between two maps $\phi = F_0$ and $\psi = F_1$ of $C$ in $P^r$. Since the 2-form $F_t^*(\phi)$ is closed, we have that $\int_{\partial(C\times[0,1])} F_t^*(\Phi) = \int_{C\times[0,1]} \mathrm{d}F_t^*(\Phi) = 0$, where $\partial$ is the boundary operator. Now $\partial(C \times [0,1]) = (C \times 0) \cup (\bar{C} \times 1)$, where $\bar{C}$ denotes $C$ with the opposite orientation. Then $0 = \int_{\partial(C\times[0,1])} F_t^*(\Phi) = \int_C \phi^*(\Phi) - \int_C \psi^*(\Phi)$, which shows that deg $(\phi) =$ deg $(\psi)$.

From the geometrical point of view, one can show that the degree of a holomorphic map $\phi : C \to \mathbb{P}^r$ is given by deg $(\phi) = b \cdot n$, where $b$ stands for the degree of the (possibly ramified) covering $C \to \phi(C)$ induced by $\phi$, and $n$ is the degree of $\phi(C)$ as subvariety of $\mathbb{P}^r$. Infact, by Wirtinger's theorem $n = \int_{\phi(C)} \Phi$. Note that, when $P^1$ models are considered, $\phi(C) = \mathbb{P}^1$ and deg $(\phi)$ coincides with $b$, which gives the number of counterimages $\phi^{-1}(p)$ of any point $p \in \mathbb{P}^1$. In the case of maps $\phi : C \to \mathbb{P}^r$ $(r > 1)$, neither $b$ nor $n$ are separately invariant under homotopies; in fact they are not invariant even under a holomorphic homotopy. Nevertheless, as we know, their product $b \cdot n =$ deg $(\phi)$ is a homotopy invariant.

The relevance of this invariant for $\mathbb{P}^r$ models is due to the following result [19];

PROPOSITION 3.5. *In any given homotopy class of maps $\phi : C \to \mathbb{P}^r$, holomorphic or anti-holomorphic maps (if any) give absolute minima of the energy functional $E$.*

*Proof.* Since both the integrals of $\epsilon^{(1,0)}$ and $\epsilon^{(0,1)}$ over $C$ are positive definite, we have that $E(\phi) \geqslant |\deg(\phi)|$. Then $E(\phi) = |\deg(\phi)|$ if and only if $\epsilon^{(1,0)} = 0$ or $\epsilon^{(0,1)} = 0$. In the first case is must be $\xi_{\bar z} = 0$ and $\phi$ is holomorphic, in the second case $\bar{\xi}_z = 0$ and $\phi$ is anti-holomorphic.

COROLLARY 3.6. *Holomorphic and anti-holomorphic maps $\phi : C \to \mathbb{P}^r$ are harmonic.*

REMARK. When $r = 1$, we recover the result of Belavin and Polyakov [11] for the standard non linear $\sigma$-model.

## 4. HARMONIC AND HOLOMORPHIC MAPS

4.1. The complex formulation of the problem given in the previous section has the advantage of yielding in a natural way plenty of classical solutions of finite energy for the $\mathbb{P}^r$ models subjected to generalized boundary conditions. Moreover, proposition 3.5 tells us that, in certain cases (see below), the full non linear field equations are indeed equivalent to the best known system of differential equations, i.e. Cauchy-Riemann equations.

The next natural step is to look for the converse of corollary 3.6. In other words, we know that holomorphic maps are classical solutions and we would like to know if any classical solution corresponds to a holomorphic map. If this were the case, one could use the great deal information available from algebraic geometry to classify classical solutions of $\mathbb{P}^r$ models and to study their parameter spaces. Unfortunately, this is not true for general $\mathbb{P}^r$ models. Nevertheless, we shall review in the following what is known; as we shall see, in turns out that for the simplest (and probably more physical) cases every classical solution of finite energy is either given by a (anti)-holomorphic map or can be constructed in terms of a (anti)-holomorphic map.

4.2. The first case to be considered is that of the standard $SO(3)$ invariant $\sigma$-model, subjected to generalized boundary conditions. We already know that classical solutions correspond to maps $\phi : C \to \mathbb{P}^1$, which are harmonic with respect to the Fubini-Study metric on $\mathbb{P}^1$ and to a given complex structure $C$ on $S_g$. We recall the following result, referring to [19] for the proof.

PROPOSITION 4.3. *Any classical solution $\phi : C \to \mathbb{P}^1$, defined on a curve $C$ of genus $g$, with* deg $(\phi) \geqslant g$ *is holomorphic.*

REMARK. Note that by changing the orientation of $C$, a holomorphic map becomes an anti-holomorphic map on $\overline{C}$. So, proposition 4.3 tells also that, if deg $(\phi) \leqslant -g$, $\phi$ is antiholomorphic. In the following we shall limit ourselves to state the relevant results for the holomorphic case. By reversing the orientation of $C$ one obtains analogous results for anti-holomorphic maps.

As a corollary to proposition 4.3, we note that for the usual form of the non linear $\sigma$-model the domain of the problem is the Riemann sphere, for which $g = 0$. Accordingly, we have

COROLLARY 4.4. *Any classical solution $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is holomorphic.*

REMARK. For the torus, the genus $g$ is equal to 1. Then every classical solution of degree larger than zero has to be holomorphic. However, there are no holomorphic maps of degree one from the torus to $\mathbb{P}^1$. Indeed any such map is invertible and the inverse would be holomorphic; hence we would have that the torus and the Riemann sphere are isomorphic as complex manifolds, which is false. In particular we see that there are no harmonic maps of degree one from the torus to $\mathbb{P}^1$, that is there are no classical solutions of degree one for the standard $\sigma$-model defined on a torus.

According to proposition 4.3, non holomorphic solutions of $\mathbb{P}^1$ models are bound to have small degrees. The following results tells us that this class is not empty. Infact we have [22]

PROPOSITION 4.3. *For any* d, $0 \leqslant \mathrm{d} \leqslant g - 1$, *there exist a complex structure $C$ on $S_g$ and a harmonic map $\phi : C \to \mathbb{P}^1$ with* deg $(\phi) = \mathrm{d}$ *which is not holomorphic* *with respect to that complex structure.*

Note that the proposition above tells us nothing about the existence of harmonic maps to $\mathbb{P}^1$ (either holomorphic or not) of small degree from a surface of genus $g$ with a *given* complex structure.

4.6. Our knowledge about the general case of $\mathbb{P}^r$ models is even less complete, except when they are defined on the Riemann sphere, that is they are subjected to the usual boundary conditions. In what follows we shall need the concept of isotropic maps. Given any map $\phi : C \to \mathbb{P}^r$, we can pull-back on $C$ the tangent bundle $T\mathbb{P}^r$ of $\mathbb{P}^r$ together with any structure defined on it. We shall denote by $\langle\, ,\rangle$ the hermitean fibre metric on this pull-back bundle which is induced by the Fubini-Study metric on $T\mathbb{P}^r$, and by $D'$, $D''$ the holomorphic and anti--holomorphic parts of the covariant differential operators obtained by pulling back the hermitean connection of $T\mathbb{P}^r$.

DEFINITION. *We say that a map $\phi : C \to \mathbb{P}^r$ is* isotropic *if, for any $j$, $k \geqslant 1$, one has $\langle D'^j \phi, D''^k \phi\rangle = 0$ at any point of $C$. ($D'^j$ and $D''^k$ stand for the iterations of $D'$ and $D''$ $j$ and $k$ times respectively).*

REMARK. It can be shown [15] that any harmonic map $\phi : \mathbb{P}^1 \to \mathbb{P}^r$ is isotropic. The same holds true for harmonic maps of non-zero degree from the torus to $\mathbb{P}^r$.

We recall also that

DEFINITION. *A map $\phi : C \to \mathbb{P}^r$ is said to be* full, *if its image $\phi(C)$ is not contained into any $\mathbb{P}^s \subset \mathbb{P}^r$ $(s < r)$.*

The concept of isotropy of maps was introduced because of the following result [16].

PROPOSITION 4.7. *There is a one-to-one correspondence between isotropic and full harmonic maps $\phi : C \to \mathbb{P}^r$ and pairs $(f, k)$ where $f : C \to \mathbb{P}^r$ is a holomorphic map and $k$ is an integer $0 \leqslant k \leqslant r$.*

REMARK. This proposition is very useful in the case of $\mathbb{P}^r$ models with standard boundary conditions, whose classical solutions are harmonic maps $\phi : \mathbb{P}^1 \to \mathbb{P}^r$. Indeed, any such map is isotropic. Then, in this case, holomorphic maps give rise to all the classical solutions. This will be further discussed in section 6.

We refer to [16] for the proof of 4.7. The most important point for our concern is to show here how a holomorphic map can generate a solution which is not holomorphic. Any full holomorphic map $f: C \to \mathbb{P}^r$ can be locally represented by a vector valued function $v: C \to \mathbb{C}^{r+1}$ which associates to any $z \in U \subset C$ the line in $\mathbb{C}^{r+1}$ corresponding to $f(z)$, i.e. $f(z) = (v_0(z), \ldots, v_r(z))$. Let now $v^{(i)}(z) = \dfrac{\mathrm{d}^i}{\mathrm{d}z^i}\, v(z)$ denote the $i$-th order derivative of $v(z)$. For any $i$, $v^{(i)}$ is still a vector valued function $v^{(i)}: C \to \mathbb{C}^{r+1}$ locally defined on $U \subset C$. From these data, one can construct a map which associates to $z \in U$ the linear span of the $v^{(i)}$'s in $\mathbb{C}^{r+1}$. Let $Gr(k+1, r+1)$ be the Grassmannian of $k+1$-planes in $\mathbb{C}^{r+1}$. The *$k$-th associated curve* of $f$ is a map $f_k: C \to Gr(k+1, r+1)$, which is locally defined by letting $f_k(z)$ be the $k+1$-plane in $\mathbb{C}^{r+1}$ spanned by $v(z), v^{(1)}(z), \ldots, v^{(k)}(z)$. We refer to [21] in order to realize that $f_k$ is well defined, in that it does not vanish identically and is independent of the choice of the local representation of $f$ by $v(z)$. Finally we note that $f_0 = f$ and that it is convenient to put $f_{-1}$ equal to the zero map. Associated curves to anti-holomorphic maps can be obviously defined in the same way.

Let now $f_k(z)^\perp$ be the orthogonal complement of $f_k(z)$ with respect to the standard hermitian metric of $\mathbb{C}^{r+1}$. The (anti-holomorphic) map $f: C \to \mathbb{P}^r$ given by $\tilde{f}(z) = f_{r-1}(z)^\perp$ is called the *polar curve* of $f$. (Recall that

$$C \xrightarrow{f_{r-1}} Gr(r, r+1) \xrightarrow{\perp} Gr(1, r+1) = \mathbb{P}^r).$$

Now for any $k$, $0 \leqslant k \leqslant r$, we can define a (non holomorphic) map $\psi_k: C \to \mathbb{P}^r$ by

$$\psi_k(z) = (f_{k-1}(z) \oplus \tilde{f}_{r-k-1}(z))^\perp.$$

To explain this definition, we note that $f_{k-1}(z)$ in an $k$-plane in $\mathbb{C}^{r+1}$, $\tilde{f}_{r-k-1}(z)$ is an $r-k$-plane in $\mathbb{C}^{r+1}$. Their direct sum is an $r$-plane in $\mathbb{C}^{r+1}$ whose orthogonal complement is a line in $\mathbb{C}^{r+1}$, i.e. a point in $\mathbb{P}^r$.

It can be shown that $\psi_k(z)$ is isotropic, but the important point for us is that it is harmonic. Infact $\psi_k$ is the composition of a map $\psi_k': C \to Gr(k, r+1) \times Gr(r-k, r+1)$ given by $\psi_k'(z) = (f_{k-1}(z), \tilde{f}_{r-k-1}(z))$ and the orthogonal projection $\pi$. Now $\psi_k'$ is harmonic, because its two components are separately harmonic ($f_{k-1}$ is holomorphic and $\tilde{f}_{r-k-1}$ is anti holomorphic), and the orthogonal projection, being a Riemannian submersion [22], fulfills the requirements in order that $\pi \circ \psi_k'$ be harmonic.

4.8. Proposition 4.7 gives us a tool for constructing all the classical solutions which are full and isotropic, starting from holomorphic maps $f: C \to \mathbb{P}^r$.

Referring to section 6 for more detaided applications to $\mathbb{P}^r$ models, we shall here review some more results which will be useful in the following.

First of all it is natural to ask about the energy and the degree of a full isotropic solution $\psi_k$ constructed from a holomorphic map $f$. It can be shown [16] that

$$E(\psi_k) = \deg(f_k) + \deg(f_{k-1})$$

and

$$\deg(\psi_k) = \deg(f_k) - \deg(f_{k-1}).$$

To compute $E(\psi_k)$ and $\deg(\psi_k)$, we need to know the degree of the curves associated to $f$. Let $\mathrm{d}_k = \deg(f_k)$; obviously $\mathrm{d}_0 = \deg(f) = \mathrm{d}$ and $\mathrm{d}_{-1} = = \deg(f_{-1}) = 0$. The higher $\mathrm{d}_k$'s are given by the Plücker formulas

$$\mathrm{d}_{k+1} - 2\,\mathrm{d}_k + \mathrm{d}_{k-1} = 2g - 2 - \beta_k,$$

where $\beta_k$ is the ramification index of $f_k$ (for more details, see [21]). Since $f_r$ is a constant map, $\mathrm{d}_r$ must vanish. Summing the recurrence relations above, one easily finds that

$$\sum_{j=0}^{r-1} (r-j)\,\beta_j = (r+1)\,\mathrm{d} + r(r+1)(g-1).$$

Plücker formulas can be explicitely solved. If the conditions $\mathrm{d}_0 = \mathrm{d}$ and $\mathrm{d}_r = 0$ are imposed, we find

$$\mathrm{d}_k = (k+1)\,\mathrm{d} + k(k+1)(g-1) - \sum_{j=0}^{k-1} (k-j)\beta_j, \quad (0 \leqslant k \leqslant r).$$

We have then proved the following

PROPOSITION 4.9. *Let $\psi_k$ be a full isotropic map of rank $k$ $(0 \leqslant k \leqslant r)$, associated to a holomorphic map $f: C \to \mathbb{P}^r$ of* $\deg(f) = \mathrm{d}$. *Then the energy and the degree of $\psi_k$ are given by*

$$E(\psi_k) = \deg(\psi_k) + 2\left[k\mathrm{d} + k(k-1)(g-1) - \sum_{j=0}^{k-2} (k-j-1)\,\beta_j\right],$$

$$\deg(\psi_k) = \mathrm{d} + 2k(g-1) - \sum_{j=0}^{k-1} \beta_j.$$

REMARK. Among the $r+1$ full isotropic solutions generated by $f$, $\psi_0$ and $\psi_r$ are respectively homorphic and antiholomorphic. Infact $\psi_0 = \tilde{\tilde{f}}$ is the polar of the polar curve of $f$ and it is not difficult to show that actually $\tilde{\tilde{f}} = f$, while

$\psi_r = \tilde{f}$, being the polar of $f$, is antiholomorphic. From the formulas above, we have that $E(\psi_0) = \deg(f) = \deg(\psi_0)$ and $E(\psi_r) = \deg(f_{r-1}) = -\deg(\psi_r)$, so that the energy coincides with (minus) the degree for (anti)-holomorphic maps, as we already know. Note that $\psi_0$ and $\psi_r$ are not homotopic, because $\deg(\psi_0) \neq \neq \deg(\psi_r)$. Also, $\tilde{f} = \psi_r$ is not in general homotopic to $\overline{f(z)}$ (i.e. to the antiinstanton solution associated to $f$), since $\deg(\psi_r) \neq -\deg(f)$ and obviously $E(\psi_r) \neq \neq E(f)$. However, if $f$ is such that $\sum_{j=0}^{r-1} \beta_j = 2d + 2r(g-1)$, $\psi_r$ is homotopic to $\overline{f(z)}$ and $E(\psi_r) = E(f)$.

4.10. The construction of proposition 4.7 yields plenty of classical solutions of $P^r$ models. It is natural now to ask about their stability. Recall that, in this context, by an unstable map it is meant an extremum of the energy functional, which is not a local minimun. In order to determine the qualitative behaviour of the energy functional near a harmonic map, we must consider its second variation.

In general a vector field along a map $\phi : M \to N$ ($M, N$ complete Riemannian manifolds) is a section $v(x)$ of the bundle $\phi^*TN$ and it defines a *variation* of $\phi$ by $\phi_t(x) = \exp_\phi(tv(x))$ (for more details see e.g. [22]). Given two vector fileds $v$ and $w$ along $\phi$, we choose a two-parameter variation $\phi_{s,t}$ such that

$$v = \left.\frac{\partial \phi_{s,t}}{s}\right|_{s=t=0}, \qquad w = \left.\frac{\partial \phi_{s,t}}{t}\right|_{s=t=0}.$$

Then the *Hessian* of the energy functional at $\phi$ is the symmetric bilinear form

$$H_\phi(v, w) = \left.\frac{\partial^2 E(\phi_{s,t})}{\partial s\, \partial t}\right|_{s=t=0}.$$

The *index* of $\phi$ is the dimension of the subsapace of the space of sections of $\phi^*TN$ on which $H$ is negative definite. A map which is an extremum for $E$ is (weakly) stable if its index vanishes, i.e. if there are no variations which lower its energy.

For our purposes we do not need such a generality but we can restrict ourselves to the case of holomorphis variations. In this case, we take a holomorphic vector field $v$ along $\phi$ and we construct a variation $\phi_s$ of $\phi$, depending on a complex parameter $s$, such that

$$\left.\frac{\partial \phi_s}{\partial \bar{s}}\right|_{s=0} = 0, \qquad \left.\frac{\partial \phi_s}{\partial s}\right|_{s=0} = v.$$

For such a $v$, the second variation is [16]

$$\left.\frac{\partial^2 E}{\partial s\, \partial s}\right|_{s=0} = -i \int_C R(\phi_s, \overline{\phi_s}, \phi_{\bar z}, \overline{\phi_{\bar z}})\, \mathrm{d}z \wedge \mathrm{d}\bar z$$

where $R( \;,\;,\;,\;)$ is the curvature tensor on $TP^r$. The point is now that the integral above is non-negative, because $P^r$ has strictly positive bisectional curvature. Hence we see that (anti)-holomorphic maps are stable.

Eells and Wood [16] (see also Din and Zakrzewsky [15]) for a direct computation) were able to show that

PROPOSITION 4.11. *A classical solution $\phi : P^1 \to P^r$ is stable if and only if it is (anti)-holomorphic.*

For higher genus the results so far obtained are not complete and are grounded on an estimate of the index by means of the Riemann-Roch theorem for holomorphic vector bundles over a curve $C$. The result [16] is that for holomorphic variations

$$\text{Index}\,(\phi) \geqslant \deg\,(\phi)\,(r+1) + r(1-g).$$

We have the

PROPOSITION 4.12. *A classical solution $\phi : C \to P^r$ of degree larger than $r(g-1)/(r+1)$ is stable if and only if it is (anti)-holomorphic.*

This proposition tells us that, at least for large degree, all the isotropic solutions are infact unstable.

## 5. CLASSIFICATION OF HOLOMORPHIC MAPS OF A CURVE INTO PROJECTIVE SPACES

5.1. The results recalled in section 4 show that a certain subclass of classical solutions of $\mathbb{P}^r$ models, with generalized boundary conditions, can be represented as holomorphic maps into $\mathbb{P}^r$ of a certain Riemann surface $C$, whose genus depends on the chosen boundary conditions. In any case, holomorphic maps play a central rôle, either because they represent instanton solutions or because they provide building blocks from which more general solutions can be constructed. Moreover, when $C = \mathbb{P}^1$, these more general solutions exhaust the whole class of classical solutions.

The next question we shall ask ourselves is, roughly speaking, «how many» classical solutions of a given degree and satisfying certain boundary conditions do we expect to exist. In view of the preceeding remarks, this question is naturally translated into the problem of classifying holomorphic maps of a Riemann surface into a projective space. Fortunately enough, quite a bit is known from algebraic geometry about this problem and our task in this section will be to give a quick review of the main results. For the sake of clearness, this will be done in mathematical terms, while applications to $\mathbb{P}^r$ models will be delayed to section 6.

5.2. We first recall that isomorphism classes of complex structures on a compact orientable surface $S_g$ of genus $g$ are parametrized by an irreducible quasi-projective [24] variety $M_g$, whose (complex) dimension is $3g - 3$, when $g \geqslant 2$ and 0 or 1, when $g = 0, 1$ respectively [21]. For a point $p \in M_g$, we have a complex structure turning $S_g$ into a curve $C$, different points corresponding to inequivalent structures. The variety $M_g$ is classically called the *moduli space* for genus $g$ curves; it is singular when $g \geqslant 2$ and its singularities all arise from curves with a non trivial automorphism group. From a «physical» point of view, one may think of $M_g$ as the variety of inequivalent conformal structures on $S_g$ (see Appendix B and section 3.2).

5.3. Let $f: C \to \mathbb{P}^r$ be a non-constant holomorphic map. If $H$ stands for the hyperplane line bundle of $\mathbb{P}^r$, $f$ determines a line bundle $L = f^*(H)$ on $C$ of degree $\mathrm{d} = \deg(f)$, plus $r + 1$ distinguished sections of $L$ gotten by pulling back the homogeneous coordinates on $\mathbb{P}^r$; these sections never vanish simultaneously. Conversely, given a line bundle $L$ of degree $\mathrm{d} > 0$ on $C$ and $r + 1$ sections with no common zeroes $s_0, \ldots, s_r$ of $L$, we can construct a non constant map $f$ from $C$ to $\mathbb{P}^r$ by setting

$$f(p) = [s_0(p), \ldots, s_r(p)].$$

The sections $s_0, \ldots, s_r$ span an $s + 1$-dimensional vector subspace $V$ of $H^0(C, L)$, i.e. a *linear series* of degree d and dimension $s$ on $C$ (a $g^s_{\mathrm{d}}$ for short) with $1 \leqslant s \leqslant r$. Clearly, $V$ is base-point-free, that is no point of $C$ is a common zero of all the elements of $V$. Thus, to give a degree d non-constant map from $C$ to $\mathbb{P}^r$ is the same as giving a degree d line bundle $L$ on $C$, a base-point-free $g^s_{\mathrm{d}}, V \subset H^0(C, L)$ $(1 \leqslant s \leqslant r)$,and $r + 1$ spanning vectors for $V$, up to homothety.

5.4. In view of these remarks, we shall first focus our attention on the classification of linear series, beginning with the classification of *complete linear series* on a fixed curve $C$ (recall that a linear series $V \subsetneq H^0(C, L)$ is said to be complete if $V$ actually coincides with $H^0(C, L)$).

The basic restrictions on the numbers $r$, d for complete $g^r_{\mathrm{d}}$'s are provided by the Riemann-Roch theorem [21], which implies that

$$r \geqslant \mathrm{d} - g,$$

with equality holding in any case if $\mathrm{d} > 2g - 2$, and by Clifford's theorem [21], which states that

$$2r \leqslant \mathrm{d}$$

if $\mathrm{d} \leqslant 2g - 2$.

Actually, Clifford's theorem also says that, if the bound $2r = \mathrm{d}$ is attained and $0 < \mathrm{d} < 2g - 2$, then $C$ has to be hyperelliptic, i.e. it can be represented as a two-sheeted branched covering of $\mathbb{P}^1$. Since genus $g$ hyperelliptic curves are parametrized by a $2g - 1$-dimensional subvariety of $M_g$, we find that a general curve of genus $g > 2$ has no $g^r_{2r}$, $0 < r < g - 1$.

5.5. This is the first example of a more general phenomenon, what happens is that, for any fixed d, the higher the maximum $r$ such that $C$ has a $g^r_{\mathrm{d}}$, the more special $C$ is. To state this precisely we first recall that for any d, line bundles of degree d on $C$ are parametrized by a a smooth $g$-dimensional complete variety $\mathrm{Pic}^{\mathrm{d}}(C)$. For any couple $r$, d of non negative integers, we then define $W^r_{\mathrm{d}}(C)$ to be the subvariety of $\mathrm{Pic}^{\mathrm{d}}(C)$ whose points correspond to degree d line bundles on $C$ such that $\dim H^0(C, L) \geqslant r + 1$. In other terms, $W^r_{\mathrm{d}}(C)$ parametrizes complete $g^{\bar{r}}_{\mathrm{d}}$'s on $C$ such that $\bar{r} \geqslant r$.

The problem now is: what is the dimension of $W^r_{\mathrm{d}}(C)$? Clearly, by the Riemann--Roch theorem, we can restrict ourselves to the case $r \geqslant \mathrm{d} - g$. Moreover,

$$W^r_{\mathrm{d}}(C) = \mathrm{Pic}^{\mathrm{d}}(C), \text{ if } r = \mathrm{d} - g,$$

$$W^r_{\mathrm{d}}(C) = \emptyset, \qquad \text{if } r > \mathrm{d} - g, \mathrm{d} > 2g - 2,$$

whereas $W^{g-1}_{2g-2}(C)$ contains precisely the canonical line bundle of $C$. This, on the one hand, disposes of the cases $g = 0, 1$; on the other hand, when $g > 1$, it shows that we may limit ourselves to the case $0 < \mathrm{d} < 2g - 2$.

There is a precise guess as to what the dimension of $W^r_{\mathrm{d}}(C)$ has to be. Infact it is not too hard to show that $W^r_{\mathrm{d}}(C)$ is a determinantal subvariety of $\mathrm{Pic}^{\mathrm{d}}(C)$ and, more precisely, it can be locally defined as the locus where a certain $n \times (\mathrm{d} + n + 1 - g)$ matrix of holomorphic functions has rank at most $\mathrm{d} + n - g - r$ (here $n$ is any large enough integer) [25, 26]. Recalling that the variety of matrices of rank at most $k$ has codimension $(n - k)\ (N - k)$ in the variety of $n \times N$ matrices, we find

$$\dim W^r_{\mathrm{d}}(C) \geqslant \rho = g - (r + 1)(g - \mathrm{d} + r), \text{ if } W^r_{\mathrm{d}}(C) \neq \emptyset.$$

At this point two questions arise. First of all, when do we have equality in this inequality and, secondly, when is $W^r_d(C)$ non empty? The answer to this last question was given independently by Kempf [25] and Kleiman-Laksov [26, 27], who proved, among other things, the following;

5.6. *Existence Theorem for Special Divisors.* $W^r_d(C) \neq \emptyset$, whenever $\rho \geqslant 0$.

As for our first question, it is clear from the second part of Clifford's theorem that the answer will depend in an essential way on the curve $C$. It was long an open problem to determine what happens for a general curve $C$ of genus $g > 1$. This issue was finally settled by Griffiths and Harris [28], who showed,

5.7. *Theorem (Brill-Noether Conjecture).* Let $C$ a general curve of genus $g$. Suppose $r \geqslant d - g$. Then

$$W^r_d(C) = \emptyset \,, \qquad \text{if } \rho < 0,$$

$$\dim W^r_d(C) = \rho \,, \quad \text{if } \rho \geqslant 0.$$

5.8. Our primary interest was in (possibly incomplete) linear series of fixed degree d and dimension $r$ on $C$. These can be parametrized by a variety $G^r_d(C)$ in such a way that the natural mapping

$$\gamma : G^r_d(C) \longrightarrow W^r_d(C),$$

gotten by associating to each $g^r_d$ on $C$ the corresponding complete series, is holomorphic (see e.g. [29]). Clearly, the fibre of $\gamma$ over $L \in W^r_d(C)$ is the Grasmannian of $r + 1$-planes in $H^0(C, L)$. The basic fact about $G^r_d(C)$, which generalizes the Brill-Noether conjecture, and was proved by Gieseker [30], is

5.9. *Theorem (Petri's Conjecture).* If $C$ is a general curve of genus $g$, then

a) when $\rho = g - (r + 1)(g - d + r) < 0$, then $G^r_d(C) = \emptyset$,
b) when $\rho \geqslant 0$, $G^r_d(C)$ is smooth of dimension $\rho$.

The new fact here is that $G^r_d(C)$ is smooth. The dimension statement is actually contained in the Brill-Noether conjecture. In fact, when $r < d - g$, by the Riemann-Roch theorem, $W^r_d(C) = \mathrm{Pic}^d(C)$, whereas a general fibre of $\gamma$ is the Grasmannian of $r + 1$-planes in a $(d - g + 1)$-dimensional vector space. Hence,

$$\dim G^r_d(C) = g + (r + 1) \{(d - g + 1) - (r + 1)\} = \rho,$$

as desired.

5.10. The existence theorem of Kempf-Kleiman-Laksov is actually more precise than the bare statement 5.6, in that it also yields an explicit formula for the fundamental class $w^r_{\rm d}$ of $W^r_{\rm d}(C)$, when this variety has the «correct» dimension $\rho$. The formula is

$$w^r_{\rm d} = \frac{r! \ldots 0!}{(g - {\rm d} + 2r)! \ldots (g - {\rm d} + r)!} \, (g - \rho)! \, w^0_\rho \, ,$$

where, of course, $r \geqslant {\rm d} - g$. In particular, when $\rho = 0$ and $C$ is general, $W^r_{\rm d}(C)$ consists of exactly

$$\frac{r! \ldots 0!}{(g - {\rm d} + 2r)! \ldots (g - {\rm d} + r)!} \, g!$$

points.

A very important result concerning the connectivity of $W^r_{\rm d}(C)$ and which, in a way, generalizes the existence theorem 5.6, has been recently obtained by Fulton and Lazarsfeld [31]. What they show is

PROPOSITION 5.11. *Assume* $\rho > 0$. *Then, for* any *curve* $C$, $W^r_{\rm d}(C)$ *(or, which is the same,* $G^r_{\rm d}(C)$*) is connected.*

REMARK. When $C$ is general, this result, coupled with Petri's conjecture 5.9, shows that $W^r_{\rm d}(C)$ is irreducible. Of course, $W^r_{\rm d}(C)$ may be very well reducible for special $C$.

Another problem which has apparently received little attention is the one of characterizing curves $C$ such that the dimension of $W^r_{\rm d}(C)$ is «larger than expected». A first result in this direction is provided by the second part of Clif-ford's theorem. Martens [32] has shown that, if there are d and $r$ such that $0 < {\rm d} \leqslant g - 1$ and $\dim W^r_{\rm d}(C) \geqslant {\rm d} - 2r$, then $C$ is hyperelliptic. The case in which the dimension of $W^r_{\rm d}(C)$ is at least ${\rm d} - 2r - 1$ has been studied by Mumford [33].

5.12. As we mentioned at the beginning of this section, our interest in linear systems stems from the fact that full holomorphic mappings of degree d from a curve $C$ into $P^r$ (up to projectivities) are in $1 - 1$ correspondence with base-point-free $g^r_{\rm d}$'s on $C$. However, the results we have mentioned later deal with the totality of $g^r_{\rm d}$'s, be they base-point-free on not. Our disregard of this fine point is justified, at least in part, by the remark that, when $C$ is general and $r \geqslant {\rm d} - g, r > 0$, the map

$$W^r_{d-1}(C) \times C \longrightarrow W^r_d(C),$$

gotten by associating to each couple $(L, p)$ the line bundle $L(p)$, cannot be onto since, by 5.7,

$$\dim W^r_{\mathrm{d}}(C) \geqslant \dim W^r_{\mathrm{d}-1}(C) + 2.$$

This means that, *when $C$ is general and $r \geqslant \mathrm{d} - g$, $r > 0$, a general point of $W^r_{\mathrm{d}}(C)$ is a $g^r_{\mathrm{d}}$ with no base points.* By contrast, when $C$ is special, $W^r_{\mathrm{d}}(C)$ may consist entirely of linear systems with a non-empty base locus; an example is provided, for instance, by $W^1_3(C)$ for an hyperelliptic curve $C$ of genus $g > 2$.

## 6. ON THE ENERGY SPECTURM AND THE SPACE OF MODULI OF $\mathbb{P}^r$ CLASSICAL SOLUTIONS

6.1. Coming back to the classical solutions of $\mathbb{P}^r$ models, we shall assume, as already mentioned in section 3.2, that $C$ is a curve of genus $g$ with a fixed complex structure. This comes from the «physical» conformal structure defined on the 2-dimensional real surface $S_g$ which is the domain of the problem.

The problem we shall deal with in this section is to study the set of smooth classical solutions of $\mathbb{P}^r$ models, by describing their energy spectrum and parameter spaces.

6.2. We shall start with stable solutions, which are (anti)-holomorphic, since in this case we can directly apply the results recalled in section 5.

We know that instanton solutions fall into (disjoint) homotopy classes. These are classified by their degree or, in physical terms, by their topological charge. For any (admissible) value of the topological charge, there is only one possible value for the energy, i.e. $E(f) = |\deg(f)|$. So the energy spectrum of these solutions is in principle known, provided one knows which values of the topological charge are admissible. We shall limit ourselves to the holomorphic case, the antiholomorphic one being obtained by a reversal of orientation. From theorem 5.9, we have that for a general curve it must be

$$\mathrm{d} \geqslant \frac{s}{s+1}\, g + s,$$

where $s$ is the dimension of the least linear subspace of $\mathbb{P}^r$ containing the image of $f$. Thus, for a general curve, there are no instantons when $\mathrm{d} < (g/2) + 1$.

6.3. Next, given an admissible d, we would like to know how many «essential» parameters will a generic instanton of degree d depend on. To this purpose, we recall that to any such solution $f : C \to \mathbb{P}^r$ there are attached a base-point-free

$g^s_{\mathrm{d}}(1 \leqslant s \leqslant r)$ and $r+1$ spanning vectors for the $g^s_{\mathrm{d}}$, determined up to homothethy. The number $s$ is the dimension of the linear subspace of $\mathbb{P}^r$ spanned by $f(C)$; thus full solutions are characterized by $s = r$. As in section 5.8, we shall denote by $G^s_{\mathrm{d}}(C)$ the space of all $g^s_{\mathrm{d}}$'s on $C$; in addition, we let $\widetilde{G}^s_{\mathrm{d}}(C)$ be the open subset consisting of base-point-free series. We shall also denote by $B^{s,r}_{\mathrm{d}}(C)$ the holomorphic bundle over $\widetilde{G}^s_{\mathrm{d}}(C)$ whose fibre over $V \in \widetilde{G}^s_{\mathrm{d}}(C)$ consists of all sets of $r+1$ spanning vectors for $V$, up to homotheties. Clearly, $B^{s,r}_{\mathrm{d}}(C)$ parametrizes instanton solutions $f: C \to \mathbb{P}^r$ such that the span of $f(C)$ has dimension $s$. The closure $B^{s,r}_{\mathrm{d}}(C)$ of $B^{s,r}_{\mathrm{d}}(C)$ in the space of all degree d instanton solutions is obtained by relaxing the conditions on the $r+1$ vectors to be chosen in $V$ to the mere requirement that they do not have common zeroes.

We now assume that $C$ is general. To compute the dimension of $B^{s,r}_{\mathrm{d}}(C)$, recall that when $\rho(s) = g - (s+1)(g - \mathrm{d} + s) < 0$, $G^s_{\mathrm{d}}(C)$ is empty; hence $B^{s,r}_{\mathrm{d}}(C)$ is also empty when $\rho(s) < 0$. When $\rho(s) = 0$, $G^s_{\mathrm{d}}(C)$ is a discrete set containing

$$n = \frac{s! \ldots 0!}{(g - \mathrm{d} + 2s)! \ldots (g - \mathrm{d} + s)!} \, g!$$

points. Accordingly, $B^{s,r}_{\mathrm{d}}(C)$ is the disjoint union of $n$ copies of the homogeneous space $PGl(r)/\Gamma$, where $\Gamma$ is the group of all linear transformations fixing $s+1$ independent vectors. Finally, when $\rho(s) > 0$, $G^s_{\mathrm{d}}(C)$, and hence $\widetilde{G}^s_{\mathrm{d}}(C)$, is a smooth connected complex manifold of dimension $\rho(s)$; thus $B^{s,r}_{\mathrm{d}}(C)$ is a smooth connected complex manifold of dimension $\rho(s) + \dim(PGl(r)/\Gamma) = \rho(s) + (r+1)(s+1) - 1$.

Turning to the full space $B^r_{\mathrm{d}}(C)$ of instanton solutions $f: C \to P^r$, for a general curve $C$ this is the disjoint union

$$B^r_{\mathrm{d}}(C) = B^{1,r}_{\mathrm{d}}(C) \cup B^{2,r}_{\mathrm{d}}(C) \cup \ldots \cup B^{r,r}_{\mathrm{d}}(C).$$

In particular, we see that $B^r_{\mathrm{d}}(C)$ has irreducible components of varying dimensions. However, as we observed, the closure of each component of $B^{s,r}_{\mathrm{d}}(C)$ intersects $B^{t,r}_{\mathrm{d}}(C)$ for every $t < s$; thus, by proposition 5.11, with the sole exception of the case when $\rho(r) = 0$, $B^r_{\mathrm{d}}(C)$ is connected.

Note that all the maps parametrized by a fibre of $B^{s,r}_{\mathrm{d}}(C)$ can be obtained one from the other by the action of $PGl(r)$ on $\mathbb{P}^r$ itself. They are all homotopic and have the same energy. However, they cannot be obtained one from the other by an action of the internal symmetry group of the model, which leaves the Lagrangian invariant and trivially sends solutions into solutions. As we know, this group is the isometry group $SU(r+1)/\mathbb{Z}_{r+1}$ of the Fubini-Study metric of $P^r$. Accordingly, we would like better to parametrize the orbits of the internal symmetry group in the parameter space $B^{s,r}_{\mathrm{d}}(C)$, i.e. to parametrize the instanton solutions up to a $SU(r+1)/\mathbb{Z}_{r+1}$ gauge transformation. To simplify matters, we shall do this only for full solutions, i.e. for $B_{\mathrm{d}}^{r,r}(C)$. From the discussion above, we have the following

6.4. THEOREM. A) *The space of full homomorphic $\mathbb{P}^r$-instantons (up to a global $SU(r+1)/\mathbb{Z}_{r+1}$ gauge transformation) is the bundle*

$$N_{\mathrm{d}}^r(C) \xrightarrow{PGl(r)/SU(r+1)} G_{\mathrm{d}}^r(C),$$

*where $N_{\mathrm{d}}^r(C)$ is the quotient bundle $B_{\mathrm{d}}^r(C)/SU(r+1)$.*

B) *Let $\rho = g-(r+1)(g-\mathrm{d}+r)$. For a general curve $C$, we then have*

i) *if $\rho < 0$, then $N_{\mathrm{d}}^r(C) = \emptyset$;*

ii) *if $\rho = 0$, $N_{\mathrm{d}}^r(C)$ is the disjoint union of*

$$\frac{r! \ldots 0!}{(g-\mathrm{d}+2r)! \ldots (g-\mathrm{d}+r)!}\, g!$$

*copies of $PGl(r)/SU(r+1)$;*

iii) *if $\rho > 0$, $N_{\mathrm{d}}^r(C)$ is a smooth connected manifold of real dimension*

$$\dim N_{\mathrm{d}}^r(C) = (r+1)(2\mathrm{d}-r+1) - 2rg - 1.$$

REMARK. Note that there are special curves for which there are «more» instantons than stated in (B), while (A) holds in any case. This depends on the structure of $G_{\mathrm{d}}^r(C)$ (see section 5).

The case of $\mathbb{P}^r$ models over $\mathbb{P}^1$ will be further discussed in the following.

6.5. Besides instantons, for $r > 1$ there are other maps at which the energy functional of $\mathbb{P}^r$ models is stationary. From proposition 4.11, we know that all these maps give saddle points for the energy, that is there are perturbations which lower their energy. Hence, at a given admissible degree, we have minimum energy solutions (i.e. instantons) and possibly higher energy unstable solutions, which are homotopic to the instantons and may be thought as their «excitations».

Our knowledge about these excitations for a general $\mathbb{P}^r$ model over a curve $C$ is far from being complete. Indeed we do know something about those, among them, which are full and isotropic. This isotropy property is a rather technical requirement, which does not have, in our opinion, any «physical» counterpart. On the other hand, it yields a whole subclass of non holomorphic solutions. Recall that, according to proposition 4.7, from any instanton $\psi_0$ we can generate

$r-1$ isotropic solutions $\psi_k$ $(0<k<r)$ which are neither holomorphic nor antiholomorphic. It is clear that each $\psi_k$ will depend on the same parameters as $\psi_0$. However, in general, the $\psi_k$'s will not have the same degree of $\psi_0$ and hance cannot be considered as excited states of $\psi_0$ itself.

What one would really like to know is the spectrum of the energy of isotropic solutions at a given degree, that is the energy spectrum of the excited states of a given instanton solution. Then, for a given degree d and enegy $E>\mathrm{d}$, one can ask for the parameter space of excited solutions. We can answer these questions only partially; nevertheless, a number of results can be proved and a qualitative description of the energy spectrum of the excited states can be given.

6.6. Recall from proposition 4.9 that the energy spectrum of full isotropic solutions is given by

$$E(\psi_k)=\deg(\psi_k)+2\{k\deg(\psi_0)+k(k+1)(g-1)-\sum_{j=0}^{k-2}(k-1-j)\beta_j\},$$

$$\deg(\psi_k)=\deg(\psi_0)+2k(g-1)-\sum_{j=0}^{k-1}\beta_j\,,$$

where $\psi_0=\tilde{\tilde{f}}$ has been identified with $f$. It is apparent that $E(\psi_k)-\deg(\psi_k)$ is even, but we do not know if any even number is actually attained. We can however prove that there are infinitely many excitations at a given degree (larger than a suitable limit) with arbitrarily high energies. The proof will be given in two steps. First we state a result which is interesting by itself, since it holds for $\mathbb{P}^r$ models over $\mathbb{P}^1$, with $r\geqslant 2$.

PROPOSITION 6.7. *For any* $|\mathrm{d}|\geqslant r-2$, *there exist full classical solutions* $\phi:\mathbb{P}^1\to\mathbb{P}^r$ $(r\geqslant 2)$ *with* $\deg(\phi)=\mathrm{d}$ *and arbitrarily high energy.*

*Proof.* We may limit ourselves to consider solutions of the form $\psi_1$, with $\deg(\psi_0)=\mathrm{d}'\geqslant r$. For $r>2$, we consider the maps $\psi_0$ obtained by projecting the Veronese map, locally given by $z\to(1,z,z^2,\dots,z^{\mathrm{d}'})$, onto $\mathbb{P}^r$ in such a way that $\psi_0$ can be locally represented by $z\to(1,z^{i_1},z^{i_2},\dots,z^{i_{r-1}},z^{d'})$ with $0<i_1<\dots<i_{r-1}<\mathrm{d}'$. These maps and their associated curves are ramified at $z=0,\infty$. According to [21], their ramification indices are given by $\beta_\ell=i_{\ell+1}-i_\ell+i_{r-\ell}-i_{r-\ell-1}-2$, where we put $i_\ell=0$ for $\ell\leqslant 0$, and $i_r=\mathrm{d}'$. Hence $\sum_{j=0}^{k-1}\beta_j=\mathrm{d}'+(i_k-i_{r-k})-2k$ and $\deg(\psi_k)=i_{r-k}-i_k$. As for $\psi_1$, we have that any value

$$-(\mathrm{d}'-2) \leqslant \deg(\psi_1) \leqslant -(r-2); \quad (r-2) \leqslant \deg(\psi_1) \leqslant (\mathrm{d}'-2),$$

can be obtained, by a suitable choice of $i_{r-1}$ and $i_1$. This shows that we have at least one isotropic solution of degree $|\mathrm{d}| \geqslant r-2$ generated by holomorphic maps of any degree $\mathrm{d}' \geqslant r$. The case $r = 2$ needs further consideration, since if $\psi_0$ is taken as above, $\deg(\psi_1) = 0$ in any case. We then consider solutions of the form $\psi_1$ generated by holomorphic maps $\psi_0 : \mathbb{P}^1 \to \mathbb{P}^2$ given locally by $z \to (1, (z+1)^{\mathrm{d}'-i}, z^{\mathrm{d}'})$, with $1 \leqslant i \leqslant \mathrm{d}'-1$. In this case $\beta_0 = i-1$, so that $\deg(\psi_1) = \mathrm{d}' - i - 1$. By considering these maps togheter with those generated by $\psi_0$, one has that also in this case any value of $\deg(\psi_1)$ within the limits given above is possible. As for the energy, we have in any case that $E(\psi_1) = \deg(\psi_1) + 2\mathrm{d}'$. Since $\mathrm{d}'$ can be arbitrarily high, $E$ can be arbitrarily large in any homotopy class.

REMARK. Incidentally, we note that any value of the energy $E = \deg(\psi_1) + 2m$, with $m \geqslant r$ is admissible in the case of $\mathbb{P}^r$ models over $\mathbb{P}^1$. In fact there exists a solution of the form $\psi_1$ with that energy and degree.

6.8. The results of proposition 6.7 can be somewhat extended to $\mathbb{P}^r$ models over a general curve $C$, by considering maps $\psi_k$ generated by quite special holomorphic maps. Infact we do not know very much about the ramification properties of holomorphic maps of $C$ into $\mathbb{P}^r$, while more can be said about maps which arise by composition as follows

$$C \xrightarrow{h} \mathbb{P}^1 \xrightarrow{f} \mathbb{P}^r.$$

Here $h$ is a branched covering of $P^1$ which, according to section 5, has degree $n = \deg(h) \geqslant [(g+1)/2] + 1$, where $[\ \cdot\ ]$ stands for the integral part.

Let $\psi_0' = f \circ h$. It is not difficult to compute the ramification indeces $\beta_k'$ of $\psi_0'$ and its associated curves; one has

$$\beta_k' = n\,\beta_k + 2(n+g-1),$$

where $\beta_k$ are the ramification indices of $f$ and its associated curves.

Accordingly, the energy and the degree of the isotropic maps $\psi_k'$ generated by $\psi_0'$ are

$$E(\psi_k') = \deg(\psi_k') + n\{E(\psi_k) - \deg(\psi_k)\} + (g-1)\{k^2(1-n) - 2k + 1 - n\} - k(k-1)n,$$

$$\deg(\psi_k') = n\deg(\psi_k) - 2\,k\,n\,g.$$

We have now the following

PROPOSITION 6.9. *Let* $\mathrm{d}' = n\mathrm{d} - 2kng$, *with* $n \geqslant [(g+1)/2] + 1$, $\mathrm{d} \geqslant r$, $0 < k < r$.

*There exist full isotropic solutions* $\phi : C \to \mathbb{P}^r$ *of degree* d$'$ *and arbitrarily high energy.*

*Proof.* If deg $(\phi) = $ d$'$, there exists in the homotopy class of $\phi$ at least one solution of the form $\psi_k'$, generated by $\psi_0' = f \circ h$. From proposition 6.7, we see that one can choose $f$ such that $E(\psi_k)$ is arbitrarily high. The result then follows from the formula above for the energy $E(\psi_k')$.

REMARK. Albeit far less complete than in the case of $\mathbb{P}^r$ models over $\mathbb{P}^1$, the proposition above leads to conjecture that the energy spectrum of a general $\mathbb{P}^r$ model is qualitatively similar to that of proposition 6.7, showing arbitrarily high energy excitations of any given instanton solution. Our proof, however, holds only for certain degrees. This is because we restricted ourselves to consider suitable composite maps, which let the proof be technically easy. To proove a proposition analogous to 6.7, one would need to classify the possible ramification behaviours of holomorphic maps $\phi : C \to \mathbb{P}^r$ and their associated curves, for a general curve $C$. Very little seems to be known in this direction.

6.10. As for the parameter spaces of full isotropic solutions, we have seen that they coincide with the parameter spaces of the holomorphic maps from which isotropic maps are generated. In principle, the question is answered by proposition 6.4, with minor modifications concerning the notion of «effective» parameters. However, such an information is of little use, since one would like to know the parameter space of solutions with a given degree and energy. It should be clear by now, that we cannot answer this question in full generality, because we do not know which energy values are admissible for excitations of a given degree. Nor do we know how many isotropic solutions generated by different holomorphic maps have the same energy and degree. Once again, to solve these questions, one needs to study in full detail the ramification properties of holomorphic maps of $C$ into $\mathbb{P}^r$. Finally, if $C$ is not $\mathbb{P}^1$ or a torus, there may be excitations which are not isotropic. About these last solutions nothing is known.

## 7. CONCLUDING REMARKS

Although most of the models considered here have no direct physical application, they have been extensively studied in the physical literature because they provide a sort of «theoretical laboratory» for testing ideas and tools in the study of non linear field theories.

This fact is clearly apparent even at the classical level, since, as we have seen,

$\mathbb{C}P^n$ models offer a nice example of the power of the algebraic geometrical methods whenever they can be applied to a field theory.

To summarize, we have seen several interesting phenomena. First of all, we have an example of a field theory in which the «space of fields» has a rich topological structure. Namely, because of the boundary conditions, continous fields in $\mathbb{C}P^n$ models fall into disjoint homotopy classes. Accordingly, we have in principle as many variational problems for the energy functional as homotopy classes of fields. This mimics the situation of Yang-Mills gauge theories on $S^4$, where one has as many variational principles for the Yang-Mills action as principal fibre bundles on $S^4$ itself, with given structure group G. Besides, one knows that isomorphism classes of such bundles are in one-to-one correspondence with homotopy classes of maps of $S^4$ into the classifying space $BG$ of the group $G$.

Next, in any given homotopy class of maps, we have

– absolute minima of the energy functional, which are given by instantons,

– saddle points of the energy functional, which are given by isotropic maps.

All instantons are algebraic objects, i.e. (anti)-holomorphic maps $\phi : C \to \mathbb{P}^r$. Also this phenomenon is common to Yang-Mills instantons over $S^4$ [4]. Moreover, both in the $\mathbb{C}P^n$ and in the Yang-Mills case, algebraic geometry tells us a great deal about the parameter spaces of instanton solutions.

Unstable excitations have no analogue in the Yang-Mills case. It is known that finite energy solutions for Yang-Mills theory on $S^4$ which are weakly stable are actually instantons, and one is lead to guess that for $G = SU(n)$ $(n > 2)$ there might exist finite energy unstable solutions of Yang-Mills field equations, in analogy with the $\mathbb{C}P^n$ models. Up to now, none of these has been found.

Finally, we mention an application of the results above to $\mathbb{C}P^2$ models [34] where some physical interpretation of isotropic solutions in given in terms of instanton-anti-instanton composite states.

## APPENDIX A: COMPLEX PROJECTIVE SPACE

A.1. Complex projective $r$-dimensional spaces, which will be denoted by $\mathbb{P}^r$, are defined as follows. Let $\mathbb{C}^* = \mathbb{C} - 0$ be the multiplicative group of non zero complex numbers. It acts in a natural way on $\mathbb{C}^{r+1} - \{0\}$ by $(\lambda, z) \to \lambda z$, where $\lambda \in \mathbb{C}^*, z \in \mathbb{C}^{r+1} - \{0\}$. Since this action is free, one can construct the space $\mathbb{P}^r$ of its orbits. Alternatively, $\mathbb{P}^r$ can be considered as the set of complex «lines» (i.e. real 2-planes) through the origin of $\mathbb{C}^{r+1}$. Thus $\mathbb{P}^r$ is seen as the basis space of the principal fibre bundle

$$\mathbb{C}^{r+1} - \{0\} \xrightarrow{\mathbb{C}^*} \mathbb{P}^r,$$

with structure group $\mathbb{C}^*$. Any point $z = (z^0, \ldots, z^r) \in \mathbb{C}^{r+1} - \{0\}$ determines a unique point in $\mathbb{P}^r$. $(z^0, \ldots, z^r)$ are called homogenous coordinates in $\mathbb{P}^r$, $(z^0, \ldots, z^r)$ and $(\lambda z^0, \ldots, \lambda z^r)$ corresponding to the same point in $\mathbb{P}^r$ for any $\lambda \in \mathbb{C}^*$.

A.2. $\mathbb{P}^r$ can be given a standard complex structure as follows. Let $U_\alpha \subset \mathbb{P}^r$ be the open set in which $z^\alpha \neq 0$ $(\alpha = 0, \ldots, r)$. We can define a local homeomorphism $\tau_\alpha : U_\alpha \to \mathbb{C}^r$ by $p \to \xi^i_\alpha = z^i(p)/z^\alpha(p)$ $(i \neq \alpha)$. The complex structure of $\mathbb{P}^r$ then arises by defining the transition functions on the intersections $U_\alpha \cap U_\beta$ to be the holomorphic maps of $\mathbb{C}^r$ into itself given by

$$\xi^i_\alpha = \xi^i_\alpha / \xi^\alpha_\beta \qquad \text{for} \quad 0 \leqslant i \leqslant r, \quad i \neq \alpha.$$

In particular $\mathbb{P}^1$ is the Riemann sphere.

A.3. $\mathbb{P}^r$ can be given also a standard Kähler structure (see e.g. [21]). Consider the closed 2-form

$$\begin{aligned}\widetilde{\Phi} &= -4i\, \partial\, \bar{\partial} \log (z^\alpha \bar{z}_\alpha) = \\ &= -4i\, \frac{z^\alpha \bar{z}_\alpha \mathrm{d}z^\beta \wedge \mathrm{d}\bar{z}_\beta - \bar{z}_\alpha \mathrm{d}z^\alpha \wedge z^\beta \mathrm{d}\bar{z}_\beta}{(z^\alpha \bar{z}_\alpha)^2}\end{aligned}$$

on $\mathbb{C}^{r+1} - \{0\}$, where $(\bar{\partial})$ $\partial$ is the (anti) holomorphic exterior differential. If $\pi$ is the projection of the standard $\mathbb{C}^*$-bundle over $\mathbb{P}^r$ (see A.1), one can show that there exists a unique globally defined 2-form $\Phi$ on $\mathbb{P}^r$ such that $\widetilde{\Phi} = \pi^*(\Phi)$. Since d and $\pi^*$ commute, we see that $\widetilde{\Phi}$ is closed and defines the standard Kähler structure on $\mathbb{P}^r$. The Kähler metric corresponding to $\Phi$ is given by

$$\tilde{h} = h_{\alpha\bar{\beta}} \mathrm{d}z^\alpha \otimes \mathrm{d}\bar{z}^\beta = \frac{z^\alpha \bar{z}_\alpha \mathrm{d}z^\beta \otimes \mathrm{d}\bar{z}_\beta - \bar{z}_\alpha \mathrm{d}z^\alpha \otimes z^\beta \mathrm{d}\bar{z}_\beta}{(z^\alpha \bar{z}_\alpha)^2}$$

which is called the Fubini-Study metric of $\mathbb{P}^r$. It is apparent that $h$ shares the same isometry group of the standard hermitean metric $\delta = \mathrm{d}z^\alpha \otimes \mathrm{d}\bar{z}_\alpha$ of $\mathbb{C}^{r+1}$, i.e. $U(n+1)$. However, only $SU(r+1)/\mathbb{Z}_{r+1}$ is the effective isometry group of the Fubini-Study metric of $\mathbb{P}^r$.

A.5. To give a local form of $\Phi$ or of $h$ on $\mathbb{P}^r$, recall first that $\partial\bar{\partial} \log (z^\alpha \bar{z}_\alpha) = \partial\bar{\partial} \log (\lambda z^\alpha \bar{\lambda} \bar{z}_\alpha)$. On $U_\alpha$, where $z^\alpha \neq 0$, let $\lambda = 1/z^\alpha$, and get

$$\Phi = -4i\, \partial\, \bar{\partial} \log (1 + \xi^i_\alpha \bar{\xi}_{\alpha i}).$$

Moreover, it is easy to prove that on $U_\alpha \cap U_\beta$, $\partial\bar{\partial} \log (1 + \xi^i_\alpha \bar{\xi}_{\alpha i}) = \partial\bar{\partial} \log (1 + \xi^i_\beta \bar{\xi}_{\beta i})$, so that $\Phi$ is globally defined. Correspondingly, we have a local expression for the Kähler metric $h$ given by

$$h|_{U_\alpha} = \frac{(1 + \xi^i_\alpha \bar{\xi}_{\alpha i})\, d\xi^k_\alpha \otimes d\bar{\xi}_{\alpha k} - \bar{\xi}_{\alpha k}\, d\xi^k_\alpha \otimes \xi^i_\alpha\, d\bar{\xi}_{\alpha i}}{(1 + \xi^i_\alpha \xi_{\alpha i})}\ .$$

A.4. $\mathbb{P}^r$ can be also considered as the basis space of a $U(1)$-bundle with total space the sphere $S^{2r+1}$ (see e.g. [35]). Let $\mathbb{C}^{r+1}$ be given the standard hermitean structure $\langle z, z\rangle = z^\alpha \bar{z}_\alpha$, and consider the unit sphere $S^{2r+1} \subset \mathbb{C}^{r+1}$ given by $z^\alpha \bar{z}_\alpha = 1$. Any point on $S^{2r+1}$ then corresponds to a point in $\mathbb{P}^r$, with homogeneous coordinates $(z^0, \ldots, z^r)$. Conversely a point on $\mathbb{P}^r$ corresponds to the set $(\lambda z^0, \ldots, \lambda z^r)$ with $\lambda\bar{\lambda} = 1$ on $S^{2r+1}$, which is an orbit of $U(1)$ on $S^{2r+1}$ itself. We have then the principal fibre bundle

$$S^{2r+1} \xrightarrow{U(1)} \mathbb{P}^r.$$

## APPENDIX B: CONFORMAL AND COMPLEX STRUCTURES ON $S_g$

B.1. Let $S_g$ be an orientable 2-dimensional surface, and let $a$ be any $C^\infty$ metric on $S_g$. It is known that there exists an atlas of isothermal coordinates $x^1, x^2$, i.e. a covering $\{U_\alpha\}$ of $S_g$ and coordinates $x^1_\alpha, x^2_\alpha$ for which

$$a|_{U_\alpha} = f^2_\alpha (dx^1_\alpha \otimes dx^1_\alpha + dx^2_\alpha \otimes dx^2_\alpha),$$

where $f^2_\alpha$ is a non vanishing $C^\infty$ function. The invariance of $a$ implies that in $U_\alpha \cap U_\beta$, $dz_\alpha = dx^1_\alpha + i\, dx^2_\alpha$ is proportional to $dz_\beta = dx^1_\beta + i\, dx^2_\beta$, if the orientation is preserved. Then $z_\alpha$ is proportional to $z_\beta$, and $S_g$ is a complex manifold.

B.2. Explicitely the complex structure $J$ is given by

$$J_\pm(\partial/\partial x^1) = \pm\, \partial/\partial x^2; \qquad J_\pm(\partial/\partial x^2) = \mp\, \partial/\partial x^1.$$

where the sign $\pm$ depends on the orientation chosen. It is apparent that for any couple of vector fields $X$, $Y$ tangent to $S_g$, one has $a(J(X), J(Y)) = a(X, Y)$. Moreover, defining $\Phi(X, Y) = a(X, J(Y))$, it is $d\Phi = 0$, so that $a$ is Kähler with respect to $J$.

B.3. If we now consider a metric $a'$ conformal to $a$, it is apparent that any isothermal covering for $a$ is also an isothermal covering for $a'$. Therefore the two metrics induce the same complex structure. The converse is also true. If $J$ is any complex structure on $S_g$, let $z_\alpha$ be the corresponding local complex coordinate. Then, for any set $\{f^2_\alpha\}$ of real non vanishing $C^\infty$ functions such that in $U_\alpha \cap U_\beta$

$$f_\beta^2 = f_\alpha^2 \,|\, \partial z_\alpha / \partial z_\beta |^2 ,$$

we can define a tensor field $a_\alpha = f_\alpha^2 \, dz_\alpha \otimes d\bar{z}_\alpha$. It is now clear that there exists a global (Kähler) metric a on $S_g$ such that $a|_{U_\alpha} = a_\alpha$. Moreover, since the $f_\alpha^2$'s are defined up to a multiplicative positive function, we have that $J$ determines a conformal class of metrics. Summing up, we have then the

PROPOSITION. *On $S_g$ there is a one to one correspondence between complex structures and conformal equivalence classes of metrics.*

## References


[1] See e.g. T. EGUCHI, P.B. GILKEY and A.J. HANSON: *Gravitation, Gauge Theory and Differential Geometry*, Phys. Rep. **66**, 213 - 393 (1980).

[2] For a review see M.F. ATIYAH: *Geometry of Yang-Mills Fields*, Lezioni Fermiane, Scuola Normale Superiore, Pisa (1979).

[3] M.F. ATIYAH and R.S. WARD: *Instantons and Algebraic Geometry*, Comm. Math. Phys. **55**, 117 - 124 (1977).

[4] M.F. ATIYAH, V.G. DRINFELD, N.J. HITCHIN, YU.I.MANIN: *Construction of Instantons*, Phys. Lett. **65A**, 185 - 187 (1978).

[5] R. PENROSE and R.S. WARD: *Twistors for Flat and Curved Space-Time*, in General Relativity and Gravitation, Ed. A. Held, Plenum Press (1980) vol. II.

[6] N.J. HITCHIN: *Poligons and Gravitons*, Math. Proc. Camb. Phys. Soc. **85**, 465 (1979).

[7] R. CATENACCI and C. REINA: *Einstein-Kähler Surfaces and Gravitational Instantons*, Gen. Rel. Grav. **14**, 255 - 77 (1982).

[8] D. MUMFORD: *An Algebro-geometrical Construction of Commuting Operators and of Solutions of the Toda Lattice Equation, Korteweg-de Vries Equation and non Linear Equations*, Int. Symp. on Algebraic Geometry - Kyoto (1977), 115 - 153.

[9] N.J. HITCHIN: *Monopoles and Geodesics*, Commun. Math. Phys., **83**, 579 - 602 (1982).

[10] A.A. BELAVIN, A.M. POLYAKOV, A.S. SCHWARTZ and YU.S.TYUPKIN: *Pseudo Particles Solutions of the Yang-Mills Equations*, Phys. Lett. **59B**, 85 (1975).

[11] A.A.BELAVIN and A.M. POLYAKOV: *Metastable States of two Dimensional Fenomagnets*, JEP Lett. 22, 245 (1975).

[12] H. HEICHER: *$SU(N)$ invariant non-linear $\sigma$-models*, Nucl. Phys. **B146**, (1979) 215 - 223.

[13] A. D'ADDA, M. LUSHER and P. DI VECCHIO: *A $1/n$ expandable series of non-linear $\sigma$-models with instantosn*, Nucl. Phys. **D146** (1979) 63 - 76.

[14] R. CATENACCI and C. REINA: *Algebraic Classification of $CP^n$ Instanton Solutions*, Lett. Math. Phys. **5** (1981), 469 - 473.

[15] A.M. DIN and W.J. ZAKRZEWSKY: *General Classical Solution of $CP^{n-1}$ models*, Nucl. Phys. **B174** (1980), 397.

[16] J. EELLS and J.C. WOOD: *Harmonic Maps from Surfaces to Complex Projective Spaces*, Adv. Math. **49**, 217 - 263 (1983).

[17] G. WOO: *Pseudoparticle Configurations in two-dimensional Ferromagnets*, J. Math. Phys. **18** (1977), 1264 - 66.

[18] M.W. HIRSCH: *Differential Topology*, GTM n. 33 - Springer-Verlag (1976).

[19] J.C. WOOD: *Harmonic Maps and Complex Analysis*, in Complex Analysis and its Application IAEA, Vienna (1976) Vol. III, 289 - 308.

[20] J. EELLS and J.H. SAMPSON: *Harmonic Mappings of Riemannian Manifolds*, Amer. J. Math. **86** (1964), 109 - 160.

[21] P. GRIFFITHS and J. HARRIS: *Principles of Algebraic Geometry*, Wiley & Sons - New York (1978).

[22] J. EELLS and J. LEMAIRE: *A report on Harmonic Maps*, Bull. London Math. Soc. **10** (1979), 1 - 68.

[23] W.J. ZAKRZEWSKY: *Classical Solutions to $CP^{n-1}$ Models and their Generalizations*, in Lecture Notes in Physics n. 151, p. 160 - 188 Springer-Verlag (1981).

[24] D. MUMFORD: *Geometric Invariant Theory*, Springer-Verlag, Berlin-Heidelberg 1965.

[25] G. KEMPE: *Schubert Methods with an Application to Algebraic Curves*, Publications of Mathematische Zentrum, Amsterdam 1971.

[26] S. KLEIMAN and D. LAKSOV: *On the Existence of Special Divisors*, Amer. J. Math. **94** (1972), 431 - 436.

[27] S. KLEIMAN and D. LAKSOV: *Another proof of the Existence of Special Divisors*, Acta Math. **132** (1974), 163 - 176.

[28] P. GRIFFITHS and J. HARRIS: *The Dimension of the Variety of Special Linear Systems on a General Curve*, Duke Math. J. **47** (1980), 233 - 272.

[29] E. ARBARELLO and M. CORNALBA: *Su una congettura di Petri*, Comment. Math. Helv. **56** (1981), 1 - 38.

[30] D. GIESEKER: *Stable Curves and Special Divisors, I.*, Invent. Math. **66**, 251 - 75.

[31] W. FULTON and R. LAZARSFELD: *On the Connectedness of Degeneracy Loci and Special Divisors*, Acta Math. **146**, 271 - 283 (1981).

[32] H. MARTENS: *On the Varieties of Special Divisors on a Curve*, J. Reine Angew. Math. **227** (1967), 111 - 120.

[33] D. MUMFORD: *Prym Varieties, I*, in Contributions to Analysis, Academic Press, New York 1974.

[34] R. CATENACCI, M. MARTELLINI, C. REINA: *On the Energy Spectrum of $CP^2$ Models*, Phys. Lett. **115B**, 461 - 2 (1982).

[35] N. STEENROOD: *The Topology of Fibre Bundles*, Princeton Univ. Press (1951).

[36] R. CATENACCI, M. CORNALBA, C. REINA: *On the Energy Spectrum and Parameter Spaces of Classical $CP^n$ Models*, Commun. Math: Phys. **89**, 375 - 386 (1983).

[37] C. REINA: *Algebraic Geometrical Methods in Field Theory: the Example of $CP^n$ models*, Proceedings of the *International Meeting on Geometry and Physics*, (1982) in press.

*Manuscript received: July 3, 1983.*

*Invited paper*